A new METHOD *of resolving* CUBIC EQUATIONS.
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THE roots of a cubic equation of this form, $x^3 + 3c.x^2 + 3c^2.x + c^3 - a = o$ which differs from a power only in its last term, can be found, by transposing, a, and extracting the cubic root on each side, provided, a, is not an impossible binomial. Read June 10th, 1797.

PROBLEM. To reduce any cubic equation to this form, $x^3 + 3c.x^2 + 3c^2.x + c^3 - a = o$, that is, to reduce it to an equation, in which, the square of the co-efficient of the second term is triple the co-efficient of the third.

IF the roots of a cubic equation, $x^3 + px^2 + qx + r = o$, are encreased or diminished by any quantity, p^2 and $3q$, will be encreased or diminished by an equal quantity, if multiplied, will be multiplied by an equal quantity, therefore their equality or inequality, not affected by those transformations.

$x =$

$$
\begin{array}{lrl}
 & & x^3 + px^2 + qx + r = 0 \\
 & x^3 = & y^3 + 3ay^2 + 3a^2y + a^3 \\
x = \overline{y+a} & px^2 = & py^2 + 2pay + pa^2 \\
 & qx = & qy + qa \\
 & r = & r
\end{array}
$$

$p^2 = 9a^2 + 6ap + p^2$ and $3q. = 9a^2 + 6ap + 3q$ therefore p^2 and $3q$. encreaſed by the ſame quantity, viz. $9a^2 + 6ap$

$$x^3 + px^2 + qx + r = 0$$

$$x = \frac{y}{a} \qquad y^3 + pay^2 + qa^2y + a^3r = 0$$

$p^2 = p^2a^2$ and $3q = 3qa^2$ therefore both multiplied by the ſame quantity, viz. a^2.

Hence it appears that the problem cannot be effected by thoſe transformations.

But the equation, $x^3 + px^2 + qx + r = 0$ by transforming the roots into their reciprocals, and freeing the firſt term from a coefficient becomes. $x^3 + qx^2 + prx + r^2 = 0$ therefore if in the propoſed equation $q^2 = 3pr$, then by transforming the roots into their reciprocals, and freeing the firſt term from a co-efficient the equation will be reduced to the required form.

Any cubic equation being propoſed, there is a quantity, by which if the roots are encreaſed (or diminiſhed) q^2 will become equal

equal $3pr$, the value of this quantity may be inveſtigated by ſolving a quadratic equation.

Thus let the equation be,

$$x^3+px^2+qx+r=0$$

$$x=\overline{y+e} \qquad \begin{array}{r} y^3+3ey^2+3e^2y+e^3 \\ py^2+2pey+pe^2 \\ qy+qe \\ +r \\ \hline \end{array}$$

$$\overline{3e^2+2pe+q}^2 = 9e^4+12pe^3 \begin{array}{c} +6q \\ +4p^2 \end{array} .\, e^2+4pqe+q^2$$

$$3\times\overline{e^3+pe^2+qe+r}\times\overline{3e+p} = 9e^4+12pe^3 \begin{array}{c} +9q \\ +3p^2 \end{array} .\, e^2 \begin{array}{c} +9r \\ +3pq \end{array} .\, e+3pr$$

$$9e^4+12pe^3 \begin{array}{c} +6q \\ +4p^2 \end{array} .\, e^2+4pqe+q^2 = 9e^4+12pe^3 \begin{array}{c} +9q \\ +3p^2 \end{array} .\, e^2 \begin{array}{c} +9r \\ +3pq \end{array} .\, e+3pr$$

$$\begin{array}{c} p^2 \\ -3q \end{array} .\, e^2 \begin{array}{c} +pq \\ -9r \end{array} .\, e \begin{array}{c} +q^2 \\ -3pr \end{array} = 0$$

Let it be required to find the roots of this equation, $x^3+6x^2+3x+2=0$. Subſtituting in the formula

$$\begin{array}{c} 36 \\ -9 \end{array} .\, e^2 \begin{array}{c} +18 \\ -18 \end{array} .\, e \begin{array}{c} +9 \\ -36 \end{array} = 0 \because 27e^2=27 \because e^2=1 \because e=1.$$

x^3

$$\begin{array}{lrr}
 & & x^3 + 6x^2 + 3x + 2 = 0 \\
 & x^3 = & y^3 + 3y^2 + 3y + 1 \\
x = \overline{y + 1} & 6x^2 = & 6y^2 + 12y + 6 \\
 & 3x = & 3y + 3 \\
 & 2 = & 2 \\
y = \frac{1}{v} & & y^3 + 9y^2 + 18y + 12 = 0 \\
v = \frac{z}{12} & & 12v^3 + 18v^2 + 9v + 1 = 0 \\
 & & z^3 + 18z^2 + 108z + 144 = 0
\end{array}$$

Extract the cubic root on each side } $z^3 + 18z^2 + 108z + 216 = 72$

$$z + 6 = \sqrt[3]{72} \therefore z = -6 + 2\sqrt[3]{9}$$

$$v = \frac{-6 + 2\sqrt[3]{9}}{12} \therefore y = \frac{12}{-6 + 2\sqrt[3]{9}} \therefore x = \frac{12}{-6 + 2\sqrt[3]{9}} + 1$$

substituting then for $2\sqrt[3]{9}$ its 3 values, $2\sqrt[3]{9}$, $\overline{-1 + \sqrt{-3}} \times \sqrt[3]{9}$, $\overline{-1 - \sqrt{-3}} \times \sqrt[3]{9}$ the roots of the proposed equation will be

$$\frac{12}{-6 + 2\sqrt[3]{9}} + 1, \quad \frac{12}{-6\overline{-1 + \sqrt{-3}}\cdot\sqrt[3]{9}} + 1, \quad \frac{12}{-6\overline{-1 - \sqrt{-3}}\cdot\sqrt[3]{9}} + 1$$

But the roots of the given equation may be found after one transformation, for the roots of the final equation are expressed in the co-efficients of the first transformed equation, and the root

root of the firſt transformed is its abſolute quantity divided by the root of the final equation.

Let the equation when its roots are encreaſed by e, be

$$y = \frac{1}{v} \qquad y^3 + py^2 + qy + r = 0$$

$$v = \frac{z}{r} \qquad rv^3 + qv^2 + pv + 1 = 0$$

$$z^3 + qz^2 + rpz + r^2 = 0$$

$$^{*}z = -\frac{q}{3} + \sqrt[3]{\frac{q^3}{27} - r^2} \qquad v = \frac{-\frac{q}{3} + \sqrt[3]{\frac{q^3}{27} - r^2}}{r} \qquad \because y = \frac{r}{-\frac{q}{3} + \sqrt[3]{\frac{q^3}{27} - r^2}}$$

When the value of e, by which the roots are to be encreaſed (or diminiſhed) is impoſſible the coefficients of the transformed equation will be impoſſible binomials $\because a = \sqrt{\frac{q^3}{27} - r^2}$ an impoſſible binome (unleſs in particular caſes the coeff. of the impoſſible part vaniſhes) hence it appears that a, and e, will be poſſible or impoſſible in the ſame caſes.

* Since the equation $z^3 + qz^2 + rpz + r^2$ may be thus expreſſed $z^3 + 3.\frac{q}{3}z^2 + 3.\frac{q^2}{9}z + \frac{q^3}{27} = \frac{q^3}{27} - r^2$.

If the roots of a cubic equation are encreafed by $-\frac{p}{3}$ the fecond term will vanifh, if by $-\frac{p}{3} \mp \sqrt{\frac{p^2}{9} - \frac{q}{3}}$ the third term will vanifh, therefore if $p^2 = 3q$ the fecond and third terms may be exterminated together, therefore the equation will have two impoffible roots; hence it appears that the equation of the required form has two impoffible roots,* confequently the value of e, by which the roots are to be encreafed will be impoffible when all the roots of the propofed equation are real ∵ a, whofe cubic root muft be computed, will be an impoffible binomial, unlefs in the particular cafes where the coefficient of the impoffible part vanifhes.

It remains to be proved that when the propofed equation has but one poffible root, the value of e, by which the roots are to be encreafed (or diminifhed) will be poffible and confequently a,

Let the roots be $\overline{-m+\sqrt{-n}}$, $\overline{-m-\sqrt{-n}}$, $-b$

$p = \overline{2m+b}$, $q = \overline{m^2+n+2bm}$, $r = \overline{bm^2+bn}$

$$\begin{array}{cccc} \overline{p^2} & \overline{+pq} & \overline{+q^2} & \\ \cdot e^2 & \cdot e & & = 0 \\ -3q & -9r & -3pr & \end{array}$$

m^2.

* That the equation of the required form has two impoffible roots, appears alfo from this, that two of the cubic roots of a, are impoffible.

$$\begin{array}{lll}
m^2.\ e^2 & +2m^3.\ e & +m^4 \\
-2bm & -4bm^2 & -2bm^3 \\
-3n & +2mn & +b^2\,m^2 \\
b^2 & +2b^2\,m & -2bmn \\
 & -8bn & +2nm^2 \\
 & & -3b^2\,n \\
 & & +n^2
\end{array}$$

Call the coefficients β, γ, δ, $\quad e = -\frac{\gamma}{2\beta} \mp \sqrt{\frac{\gamma^2}{4\beta^2} - \frac{\beta\delta}{\beta^2}}$

therefore it is to be proved that $\overline{\frac{\gamma^2}{4} - \beta\delta}$ is affirmative, and confequently the fquare root poffible.

$$\frac{\gamma^2}{4} = \overline{m^3 - 2bm^2 + mn + b^2m - 4bn}^{\,2} =$$

$$\begin{array}{lllllll}
m^6 & -4bm^5 & +2n.\ m^4 & -12bn.\ m^3 & +18b^2n.\ m^2 & -8bn^2.\ m & +16b^2n^2 \\
 & & +6b^2 & -4b^3 & +\ n^2 & -8b^3n & \\
 & & & & +\ b^4 & &
\end{array}$$

$$\beta\delta = \overline{m_4 - 2bm^3 + b^2m^2 - 2bmn + 2nm^2 - 3b^2n + n^2} \times \overline{m^2 - 2bm - 3n + b^2} =$$

$$\begin{array}{lllllll}
m^6 & -4bm^5 & +6b^2.m^4 & -4b^3.m^3 & -5n^2.m^2 & +4bn^2.m & +10b^2n^2 \\
 & & -n & & +\ b^4. & +4b^3n & -3n^3 \\
 & & & & & & -3b^4n
\end{array}$$

$$\begin{array}{llllll}
\frac{\gamma^2}{4} - \beta\delta = & 3n.\ m^4 & -12bn.\ m^3 & +18b^2n.\ m^2 & -12bn^2.m & +6b^2n^2 \\
 & & & +\ 6n^2 & -12b^3n & +3n^3 \\
 & & & & & +3b^4n
\end{array}$$

but this remainder may be refolved into thofe parts.

$$\begin{array}{rcl} \overline{m^4 - 4bm^3 + 6b^2m^2 - 4b^3m + b^4} \times 3n & = & \overline{m-b}.^4\ 3n \\ \overline{m^2 - 2bm + b^2} \times 6n^2 & = & \overline{m-b}^2 .\ 6n^2 \\ +3n^3 & = & 3n^3 \end{array}$$

n, is affirmative, and $\overline{m-b}^4$ and $\overline{m-b}^2$ alfo affirmative therefore the remainder affirmative.

Let the roots be $-m\ -m-b$

$$\begin{array}{llll} m^2. & e^2 + m^2.\ 2m.\ e + & m^2\ .\ m^2 & \\ -2bm & -2bm & -2bm & = o \\ +b^2 & +\ b^2 & +\ b^2 & \end{array}$$

$$\overline{e^2 + 2m.\ e + m^2 = o}$$

$$e = -m \mp \sqrt{m^2 - m^2}$$

therefore when two roots of the propofed equation are equal, the value e, by which the roots are to be encreafed will be one of the equal roots, therefore the two laft terms of the transformed equation will vanifh, therefore reducible to a fimple equation, which will give the remaining root *e.g.* $x^3 + 7x^2 + 16x + 12 = o$

$$\left.\begin{array}{c} 49 \\ -48 \end{array}\right. .\ e^2 \begin{array}{c} +112 \\ -108 \end{array} .\ e \begin{array}{c} +256 \\ -252 \end{array} = o \therefore e^2 + 4e + 4 = o \therefore e = -2 \mp \sqrt{4-4}$$

$x^3 =$

$$
\begin{array}{lllr}
 & x^3 & = & y^3 - 6y^2 + 12y - 8 \\
x = y + 2 & 7x^2 & & 7y^2 - 28y + 28 \\
 & 16x & & 16y - 32 \\
 & 12 & & + 12 \\
\hline
 & & & y^3 + y^2 \quad . \quad . = 0
\end{array}
$$

$$\because y = -1 \because x = -3, -2, -2.$$